Limiting Amplitude Principle for a Hyperbolic Metamaterial in Free Space

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Abstract. Harmonic wave propagation in hyperbolic metamaterials is described by the Maxwell equations with a frequency-dependent tensor of dielectric permittivity. For a range of frequencies, this tensor has eigenvalues of opposite signs, and thus, in two dimensions, the harmonic Maxwell equations can be written as a Klein-Gordon equation. This technical report is mainly dedicated to the proof of the limiting amplitude principle for the simplest case of such a problem, and is a companion to the manuscript [8].

1 Introduction The beginning (Sections 1.2-1.3) of this report is not new, and follows almost verbatim [8]. Appendix A is taken from [8]. The outline of the report can be found in Section 1.5.

1.1 The model Probably the simplest model of a hyperbolic metamaterial is provided by a 2D strongly magnetized cold plasma model, cf. [15, 4]. In the system of units chosen so that the speed of light c, dielectric permittivity ε_0 and magnetic permeability μ_0 of vacuum satisfy $c = \varepsilon_0 = \mu_0$, this model reads

$$\begin{aligned}
&\partial_t E_x - \partial_y H_z = 0, \\
&\partial_t E_y + \partial_x H_z + j = 0, \qquad \partial_t j - \omega_p^2 E_y = 0, \\
&\partial_t H_z + \partial_x E_y - \partial_y E_x = 0, \qquad (\mathbf{x}, t) \equiv (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}.
\end{aligned} \tag{1.1}$$

Here $\boldsymbol{E}$ is an electric field, H_z is a magnetic field, j is a current and $\omega_p > 0$ is a plasma frequency.

Let us introduce, for $u, v \in L^2(\mathbb{R}^2)$:

$$(u, v) = \int_{\mathbb{R}^2} u\bar{v}\, d\boldsymbol{x}, \qquad \|u\|^2 := \int_{\mathbb{R}^2} |u|^2 d\boldsymbol{x}.$$

The following energy of (1.1) is conserved:

$$\frac{d}{dt}\mathcal{E}(t) = 0, \qquad \mathcal{E}(t) = \frac{1}{2}\left(\|E_x(t)\|^2 + \|E_y(t)\|^2 + \|H_z(t)\|^2 + \omega_p^{-2}\|j(t)\|^2\right).$$

Using energy techniques, it is possible to show that the initial-value problem for (1.1) is well-posed.

In order to see the relation between the above problem and the hyperbolic effects in wave propagation, let us rewrite it in the harmonic regime. We apply to (1.1) the Fourier-Laplace transform, defined for causal functions of polynomial growth by

$$\hat{u}(\omega) = \int_0^\infty e^{i\omega t} u(t) dt, \qquad \omega \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}/ \tag{1.2}$$

Let us introduce $\mathbf{curl} = (\partial_y, -\partial_x)^T$, $\operatorname{curl} \boldsymbol{v} = \partial_x v_y - \partial_y v_x$, and $\underline{\underline{\varepsilon}}(\omega) = \operatorname{diag}(1, \varepsilon(\omega))$, where

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \tag{1.3}$$

This yields the following problem:

$$- i\omega \underline{\underline{\varepsilon}}(\omega)\hat{\mathbf{E}} - \mathbf{curl}\hat{H}_z = 0, \tag{1.4}$$

$$- i\omega \hat{H}_z + \operatorname{curl} \hat{\mathbf{E}} = 0, \tag{1.5}$$

For $\omega \in (0, \omega_p)$, $\varepsilon(\omega) < 0$, hence the above model defines a hyperbolic metamaterial [12]. Rewriting the problem for $\hat{H}_z$ yields

$$\omega^2 \hat{H}_z + \varepsilon(\omega)^{-1}\partial_x^2 \hat{H}_z + \partial_y^2 \hat{H}_z = 0, \qquad (x, y) \in \mathbb{R}^2. \tag{1.6}$$

We also introduce for brevity

$$\mathcal{L}_\omega u := \omega^2 u + \varepsilon(\omega)^{-1}\partial_x^2 u + \partial_y^2 u. \tag{1.7}$$

We see that when $0 < \omega < \omega_p$, $\varepsilon(\omega) < 0$, and the operator $\mathcal{L}_\omega$ becomes hyperbolic.

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1.2 Preliminaries: Fourier transform, weighted Sobolev spaces We define, for $u \in L^1(\mathbb{R}^2)$, s.t. $\hat{u} \in L^1(\mathbb{R}^2)$, its partial and full Fourier transforms:

$$\mathcal{F}_x u(\xi_x, y) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \mathrm{e}^{i\xi_x x'} u(x', y) dx', \quad \mathcal{F}_y u(x, \xi_y) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \mathrm{e}^{i\xi_y y'} u(x, y') dy',$$
$$\mathcal{F}u(\xi_x, \xi_y) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \mathrm{e}^{i\boldsymbol{\xi}\cdot\mathbf{x}} u(x, y) dx dy, \quad \mathcal{F}^{-1}\hat{u}(x, y) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \mathrm{e}^{-i\boldsymbol{\xi}\cdot\mathbf{x}} \hat{u}(\xi_x, \xi_y) d\xi_x \, d\xi_y.$$

First of all, let us define isotropic weighted Sobolev spaces:

$$L^2_s(\mathbb{R}^2) \equiv L^2_s := \{v \in L^2_{loc}(\mathbb{R}^2) : \int\limits_{\mathbb{R}^2} (1 + x^2 + y^2)^s |v(x, y)|^2 dx\, dy < \infty\},$$
$$H^1_s(\mathbb{R}^2) \equiv H^1_s := \{v \in L^2_s(\mathbb{R}^2) : \nabla v \in (L^2_s(\mathbb{R}^2))^2\}.$$

Weighted anisotropic Sobolev spaces are defined similarly:

$$L^2_{s,\perp}(\mathbb{R}^2) \equiv L^2_{s,\perp} := \{v \in L^2_{loc}(\mathbb{R}^2) : \int\limits_{\mathbb{R}^2} (1 + y^2)^s |v(x, y)|^2 dx\, dy < \infty\},$$
$$H^1_{s,\perp}(\mathbb{R}^2) \equiv H^1_{s,\perp} := \{v \in L^2_s(\mathbb{R}^2) : \nabla v \in (L^2_{s,\perp}(\mathbb{R}^2))^2\}.$$

with the norm

$$\|v\|^2_{L^2_{s,\perp}} \equiv \|v\|^2_{s,\perp} := \int\limits_{\mathbb{R}^2} (1 + y^2)^s |v(x, y)|^2 dx\, dy.$$

Remark that for any $v \in L^2_{s,\perp}(\mathbb{R}^2)$, $v(., y) \in L^2(\mathbb{R})$. Therefore, equivalent norms on $L^2_{s,\perp}(\mathbb{R}^2)$, $H^1_{s,\perp}(\mathbb{R}^2)$ can be rewritten using the Plancherel theorem in the following form:

$$\|v\|^2_{s,\perp} = \|\mathcal{F}_x v\|^2_{s,\perp} = \int\limits_{\mathbb{R}^2} (1 + y^2)^s |\mathcal{F}_x v(\xi_x, y)|^2 d\xi_x dy, \tag{1.8}$$

$$\begin{aligned} \|v\|^2_{H^1_{s,\perp}} &= \int\limits_{\mathbb{R}^2} (1 + y^2)^s (1 + \xi_x^2) |\mathcal{F}_x v(\xi_x, y)|^2 d\xi_x dy \\ &+ \int\limits_{\mathbb{R}^2} (1 + y^2)^s |\partial_y \mathcal{F}_x v(\xi_x, y)|^2 d\xi_x dy. \end{aligned} \tag{1.9}$$

We will use the notation $a \lesssim b$ (resp. $a \gtrsim b$) to indicate that there exists $C > 0$ that may depend on ω_p and ω, s.t. $a \leq Cb$ (resp. $a \geq Cb$).

1.3 Some additional results about the problem (1.6)

1.3.1 The problem (1.6) for $\omega \in \mathbb{C} \setminus \mathbb{R}$

The following result is based on the Lax-Migram lemma.

Lemma 1.1. *Let $\omega \in \mathbb{C} \setminus \mathbb{R}$. Thus, for all $f \in H^{-1}(\mathbb{R}^2)$, there exists a unique $u_\omega \in H^1(\mathbb{R}^2)$ that satisfies*

$$\mathcal{L}_\omega u_\omega = f \quad in \mathcal{D}'(\mathbb{R}^2). \tag{1.10}$$

Moreover, $\|u_\omega\|_{H^1} \leq c|\omega_i|^{-1} \max(\omega_i^{-2}, 1)|\omega|\|f\|_{H^{-1}}$.

The unique solution to (1.10) is then given by the convolution of the source f with the fundamental solution $\mathcal{G}_\omega$:

$$u_\omega = \mathcal{N}_\omega f := \int\limits_{\mathbb{R}^2} \mathcal{G}_\omega(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d\mathbf{x}'. \tag{1.11}$$

To provide an explicit form of $\mathcal{G}_\omega$, we need the following convention.

Remark 1. *In what follows, we will use the following convention: for a complex number $z \in \mathbb{C}$, $\sqrt{z}$ denotes the principal branch of the square root, i.e.* $\operatorname{Re}\sqrt{z} > 0$ *for all* $z \in \mathbb{C}\backslash(-\infty, 0]$*; respectively,* $\log z = \log|z| + i \operatorname{Arg} z$, $\operatorname{Arg} z \in (-\pi, \pi)$.

The fundamental solution for (1.10) is given by

$$\mathcal{G}_\omega(\mathbf{x}) = \frac{-i\sqrt{\varepsilon(\omega)}}{4}\begin{cases} H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Re}\omega > 0, \operatorname{Im}\omega > 0, \\ H_0^{(2)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Re}\omega > 0, \operatorname{Im}\omega < 0, \end{cases} \tag{1.12}$$

where $H_0^{(1)}, H_0^{(2)}$ are Hankel functions of the first and second kind.

It can be shown that the partial Fourier transform of $\mathcal{G}_\omega$ is given by

$$\mathcal{F}_x\mathcal{G}_\omega = \frac{e^{i\kappa(\xi_x,\omega)|y|}}{2i\sqrt{2\pi}\kappa(\xi_x,\omega)}, \text{ with } \kappa(\xi_x,\omega) = \sqrt{-\varepsilon^{-1}(\omega)\xi_x^2+\omega^2}. \tag{1.13}$$

Let us now provide some auxiliary results about the asymptotic behaviour of $H_0^{(1)}(z)$. First, we consider behaviour for $|z| \to +\infty$, [10, pp. 266-267]. Let $z \in \mathbb{C}$ be s.t. $0 \leq \operatorname{Arg} z \leq \frac{\pi}{2}$. Then, as $|z| \to +\infty$,

$$H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{iz-i\frac{\pi}{4}}\left(1+\eta(z)\right), \qquad |\eta(z)| \leq C|z|^{-1}, \quad C > 0. \tag{1.14}$$

For $|z| \to 0$, according to [1, §9.1.3, §9.1.13]:

$$\begin{aligned} &H_0^{(1)}(z) = J_0(z) + iY_0(z), \\ &J_0(z) = 1 + g_J(z^2), \quad Y_0(z) = \frac{2}{\pi}J_0(z)\log\frac{z}{2} + g_Y(z^2), \end{aligned} \tag{1.15}$$

where g_J, g_Y are entire functions. Moreover, $g_J(0) = 0$, $g_J'(0) \neq 0$.

1.3.2 The problem (1.6) for $\omega \in (0, \omega_p)$ Let us introduce the outgoing fundamental solution, associated with the problem (1.6) for $\omega \in (0, \omega_p)$, cf. [8]:

$$\mathcal{G}_\omega^+(x,y) = \frac{1}{4\alpha}\begin{cases} H_0^{(1)}(\omega\sqrt{y^2-\alpha^{-2}x^2}), & (x,y) \in \mathcal{C}_p, \\ H_0^{(1)}(i\omega\sqrt{\alpha^{-2}x^2-y^2}), & (x,y) \in \mathcal{C}_e, \end{cases} \tag{FS}$$

$$\mathcal{C}_p = \{(x,y) \in \mathbb{R}^2_* : |y| > \alpha^{-1}|x|\}, \quad \mathcal{C}_e = \{(x,y) \in \mathbb{R}^2_* : |y| < \alpha^{-1}|x|\}. \tag{C}$$

Let, additionally, for $f \in L^2_{comp}(\mathbb{R}^2)$,

$$\mathcal{N}_\omega^+ f := \mathcal{G}_\omega * f. \tag{1.16}$$

It can be shown that

Lemma 1.2. *For all* $s, s' > \frac{1}{2}$, $\mathcal{N}_\omega^+ \in \mathcal{L}(L^2_s, H^1_{-s',\perp})$.

Moreover,

Theorem 1.3. *For all* $s, s' > \frac{1}{2}$, $f \in L^2_{s,\perp}$*, the function* $u_\omega^+ := \mathcal{N}_\omega^+ f$ *solves* $\mathcal{L}_\omega u^+ = f$.

One can verify that the solution u_ω^+ is a limiting absorption solution. This is shown in the following theorem proven in [8](Theorem 4.1), and whose proof we give for convenience of the reader in Appendix A.

Theorem 1.4. *Let* $s, s' > \frac{3}{2}$, $0 < \omega < \omega_p$*. Let* $\omega_n \in \mathbb{C}^+$, $\operatorname{Re}\omega_n > 0$*, and* $\omega_n \to \omega$ *as* $n \to +\infty$*. Then, for all* $f \in L^2_{s,\perp}$,

$$\mathcal{N}_{\omega_n} f \to \mathcal{N}_\omega^+ f \text{ in } H^1_{-s',\perp}(\mathbb{R}^2).$$

1.4 The principal result of the report The principal result of this report reads.

THEOREM 1.5. *Let $s, s' > \frac{3}{2}$. Let $f \in L^2_s(\mathbb{R}^2)$, and $0 \le \omega \le \omega_p$. Let $(\boldsymbol{E}, H_z, j)$ solve*

$$\begin{aligned}
&\partial_t E_x - \partial_y H_z = 0,\\
&\partial_t E_y + \partial_x H_z + j = 0, \quad \partial_t j - \omega_p^2 E_y = 0,\\
&\partial_t H_z + \partial_x E_y - \partial_y E_x = f e^{i\omega t},\\
&H_z(0) = E_x(0) = E_y(0) = j(0) = 0.
\end{aligned}$$

Then, as $t \to +\infty$,

$$\|H_z - h_z e^{i\omega t}\|_{L^2_{-s'}} \to 0,$$

where h_z is defined as follows:

- *for $0 < \omega < \omega_p$, $h_z = -i\omega u^+_\omega$, where $u^+_\omega = \mathcal{N}^+_\omega f$, cf. (1.16).*
- *for $\omega = \omega_p$, $h_z = 0$.*
- *for $\omega = 0$, $h_z = \frac{i\omega_p}{2\pi} \int\limits_{\mathbb{R}^2} K_0(\omega_p|x - x'|) f(x', y) dx'dy'$.*

REMARK 2. *In the above vanishing initial conditions (1.5) are taken for convenience; it is possible to show that the result holds also for sufficiently regular non-vanishing initial conditions.*

REMARK 3. *One could wonder whether*

- *the result holds in the elliptic case (i.e. when $\omega > \omega_p$);*
- *the result (up to the definition of H_f) holds when the right hand side in the equations for E_x, E_y, j does not vanish and equals $(f_x, f_y, f_j)^T e^{i\omega t}$, with $f_x, f_y, f_j \in C_0^\infty(\mathbb{R}^2)$;*
- *the harmonic behaviour as $t \to +\infty$ is true for E_x, E_y, j.*

The answer to all of these questions is positive. Because the elliptic case seems somewhat more classical, we omit it here.

In the case when the source term does not vanish in the equations involving $\partial_t E_x, \partial_t E_y, \partial_t j$, the respective proofs for the cases $\omega \notin \{\pm\omega_p, 0\}$ are a trivial extension of the proof of Theorem 1.5. However, the proofs for the cases $\omega \in \{\pm\omega_p, 0\}$ are significantly more technical (and, as a result, quite cumbersome), hence we decided to omit treating them.

In our proofs, we will follow the classical work by Eidus [9], adapted to first-order systems in the PhD thesis [5] and the work in preparation [6].

1.5 Outline of the report In Section 2, we perform the spectral analysis of the system (1.1), by rewriting it as a Schrödinger equation and using explicit expressions for its resolvent. In 3 we state limiting absorption principle for the resolvent in the operator norm topology. In 4 we prove that the resolvent is Hölder regular. In 5 we prove the limit amplitude principle (Theorem 1.5).

2 Spectral analysis of (1.1) Let us rewrite (1.1) as a generalized Schrödinger equation, following [7]. For this let us introduce

$$\mathcal{U} = \begin{pmatrix} E_x \\ E_y \\ H_z \\ \omega_p^{-1} j \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 0 & -i\partial_y & 0 \\ 0 & 0 & i\partial_x & i\omega_p \\ -i\partial_y & i\partial_x & 0 & 0 \\ 0 & -i\omega_p & 0 & 0 \end{pmatrix}. \tag{2.1}$$

The system (1.1), equipped with initial conditions $\mathcal{U}_0$ and the source $i\mathcal{F}$ can be rewritten as follows:

$$-i\frac{d\mathcal{U}}{dt} = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t), \quad \mathcal{U}(0) = \mathcal{U}_0. \tag{2.2}$$

Let us introduce as well

$$\mathcal{H} := \left(L^2(\mathbb{R}^2)\right)^2 \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2).$$

It is easy to verify that the operator $\mathcal{A}: D(\mathcal{A}) \to \mathcal{H}$, with

$$\begin{aligned}
&D(\mathcal{A}) = \mathbf{H}_{\mathrm{curl}_\perp}(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2),\\
&\mathbf{H}_{\mathrm{curl}_\perp}(\mathbb{R}^2) = \left\{\mathbf{v} \in \left(L^2(\mathbb{R}^2)\right)^2 : \partial_x v_y - \partial_y v_x \in L^2(\mathbb{R}^2)\right\},
\end{aligned}$$

is self-adjoint on $\mathcal{H}$.

The proof of the limiting absorption principle relies on the study of the spectrum of the operator $\mathcal{A}$, and, in particular, on a decomposition of its spectral measure. We start by studying the spectrum of the operator $\mathcal{A}$.

The principal result of this section is that the spectrum of $\mathcal{A}$ is purely absolutely continuous.

THEOREM 2.1 (Spectrum of $\mathcal{A}$). $\sigma(\mathcal{A}) = \sigma_{ac}(\mathcal{A}) = \mathbb{R}$.

2.1 Fourier analysis of $\mathcal{A}$ and the proof that $\sigma_p(\mathcal{A}) = \emptyset$ Extending the definition of the Fourier transform to the vector-valued distributions by

$$(\mathcal{F}\boldsymbol{v})_j = \mathcal{F}\boldsymbol{v}_j, \quad j = 1, \ldots, 4, \quad \boldsymbol{v} \in \mathcal{H},$$

we see that $\mathcal{A}$ is unitary equivalent to the matrix-vector multiplication operator $\mathbb{A}$:

$$\mathcal{A} = \mathcal{F}^{-1}\mathbb{A}\mathcal{F}, \quad (\mathbb{A}\boldsymbol{v})(\boldsymbol{k}) = \underbrace{\begin{pmatrix} 0 & 0 & k_y & 0 \\ 0 & 0 & -k_x & i\omega_p \\ k_y & -k_x & 0 & 0 \\ 0 & -i\omega_p & 0 & 0 \end{pmatrix}}_{:=\mathbb{A}(\boldsymbol{k})} \boldsymbol{v}(\boldsymbol{k}), \quad \boldsymbol{v} \in \mathcal{H}. \tag{2.3}$$

Thanks to the unitary equivalence, $\sigma(\mathcal{A}) = \sigma(\mathbb{A})$, $\sigma_p(\mathcal{A}) = \sigma_p(\mathbb{A})$; it thus remains to find the spectrum of $\mathbb{A}$. The space $\mathcal{H}$ is isometrically isomorphic to the space $L^2((\mathbb{R}^2, \nu); \mathbb{C}^4)$, with ν standing for the Lebesgue's measure; the latter space is in its turn isometrically isomorphic [13, p. 280] to the direct integral with respect to the Lebesgue measure ν of the fiber spaces $\mathbb{C}^4$ (these spaces are equipped with the Eucledian scalar product), i.e. $\mathcal{H} \simeq \mathcal{H}^{\oplus} = \int_{\mathbb{R}^2}^{\oplus} \mathbb{C}^4 d\boldsymbol{k}$. The operator $\mathbb{A} : \mathcal{H}^{\oplus} \to \mathcal{H}^{\oplus}$ (with an obvious abuse of notation) then has the form, cf. [13, p. 283],

$$\mathbb{A} = \int_{\mathbb{R}} \mathbb{A}(\boldsymbol{k}) d\boldsymbol{k}.$$

By [13, Theorem XIII.85(d, e)],

$$\lambda \in \sigma(\mathbb{A}) \iff \nu\{\boldsymbol{k} : \sigma(\mathbb{A}(\boldsymbol{k})) \cap (\lambda - \varepsilon, \lambda + \varepsilon)\} > 0 \text{ for all } \varepsilon > 0, \tag{2.4}$$

$$\lambda \in \sigma_p(\mathbb{A}) \iff \nu\{\boldsymbol{k} : \lambda \in \sigma_p(\mathbb{A}(\boldsymbol{k}))\} > 0. \tag{2.5}$$

The spectrum of every matrix $\mathbb{A}(\boldsymbol{k})$ is given by the set of the solutions λ of the characteristic equation

$$\lambda^4 - \lambda^2(|\mathbf{k}|^2 + \omega_p^2) + k_y^2\omega_p^2 = 0. \tag{2.6}$$

The solutions to the above had been studied in particular in [4]. We summarize these results in the following lemma.

LEMMA 2.2. *The solutions to (2.6) are given by continuous functions $\pm\lambda_+(\mathbf{k})$ and $\pm\lambda_-(\mathbf{k})$, defined by*

$$\lambda_{\pm}(\mathbf{k}) = \frac{\omega_p^2 + |\mathbf{k}|^2 \pm \sqrt{\Delta(\mathbf{k})}}{2}, \quad \Delta(\mathbf{k}) = \left(\omega_p^2 + |\mathbf{k}|^2\right)^2 - 4k_y^2\omega_p^2.$$

These functions satisfy: $\operatorname{Im} \lambda_-(\mathbb{R}^2) = [0, \omega_p]$, $\operatorname{Im} \lambda_+(\mathbb{R}^2) = [\omega_p, \infty)$.

Proof. First of all, let us show that for each $\lambda \in \mathbb{R}$, there exists $\mathbf{k} \in \mathbb{R}^2$ such that (2.6) holds true. This will imply that $\operatorname{Im} \lambda_-(\mathbb{R}^2) \cup \operatorname{Im} \lambda_+(\mathbb{R}^2) = \mathbb{R}$. The latter however is easy to see: for $\lambda \in \mathbb{R}$ fixed any $\mathbf{k} \in \mathbb{R}^2$ satisfying

$$\begin{aligned} &k_x^2 = \varepsilon(\lambda)(\lambda^2 - k_y^2) \quad \text{when } \lambda \neq 0, \\ &k_y = 0 \text{ when } \lambda = 0, \end{aligned} \tag{2.7}$$

satisfies (2.6). Let us remark that the set (k_x, k_y) defined by the above two equations is non-empty for any $\lambda \in \mathbb{R}$. We thus conclude $\operatorname{Im} \lambda_-(\mathbb{R}^2) \cup \operatorname{Im} \lambda_+(\mathbb{R}^2) = \mathbb{R}$.

Next, by Section 3.2 of [4], see also Lemma 3 in the same article, we know that

$$\lambda_-(k_x, k_y) \leq \omega_p \leq \lambda_+(k_x, k_y),$$

with equality signs achieved only if $k_x = 0$ and $k_y = \pm\omega_p$. Hence $\operatorname{Im} \lambda_-(\mathbb{R}^2) = [0, \omega_p]$, $\operatorname{Im} \lambda_+(\mathbb{R}^2) = [\omega_p, \infty)$. ∎

The above lemma leads to the following.

LEMMA 2.3. *$\sigma(\mathbb{A}) = \mathbb{R}$, and $\sigma_p(\mathbb{A}) = \emptyset$.*

Proof. Proof that $\sigma_p(\mathbb{A}) = \emptyset$. We will use (2.5). Suppose $\lambda \in \sigma_p(\mathbb{A})$; necessarily, all $\mathbf{k}$, for which $\lambda \in \mathbb{A}(\mathbf{k})$, lie on the curve (2.7). Thus $\mathbf{k}$ s.t. $\lambda \in \sigma_p(\mathbb{A}(\mathbf{k}))$ belong to the set of Lebesgue's measure zero. Hence, by (2.5), $\sigma_p(\mathbb{A}) = \emptyset$.

Proof that $\sigma(\mathbb{A}) = \mathbb{R}$. Let us prove that $\sigma(\mathbb{A}) = \mathbb{R}$, again by using (2.4). Let $\lambda \in \mathbb{R}$ be fixed.

Let us first consider the case $0 < \lambda < \omega_p$. By Lemma 2.2, the set

$$S_{\mathbf{k},\epsilon} = \{\mathbf{k} : \ \sigma(\mathbb{A}(\mathbf{k})) \cap (\lambda - \epsilon, \lambda + \epsilon)\}$$

is non-empty, and, for sufficiently small ϵ, by Lemma 2.2, is defined by

$$S_{\mathbf{k},\epsilon} = \lambda_-^{-1}((\lambda - \epsilon, \lambda + \epsilon)).$$

Because λ_- is a continuous function, for all ϵ sufficiently small, the set $S_{\mathbf{k},\epsilon}$ is open, and thus its Lebesgue measure is non-zero. With (2.4) we conclude that $(0, \omega_p) \in \sigma(\mathbb{A})$.

Similarly we show that $(-\infty, -\omega_p) \cup (-\omega_p, 0) \cup (\omega_p, +\infty) \subset \sigma(\mathbb{A})$. Then the points $\{0, \pm\omega_p\}$ also belong to $\sigma(\mathbb{A})$, because the spectrum is closed. □

An immediate corollary of the above and the unitary equivalence (2.3) reads

COROLLARY 2.4. *$\sigma(\mathcal{A}) = \mathbb{R}$ and $\sigma_p(\mathcal{A}) = \emptyset$.*

The above results are sufficient to prove the first part of Theorem 2.1, namely, $\sigma(\mathcal{A}) = \mathbb{R}$. In what follows we will adhere to the terminology and notation of the monograph by Schmüdgen [14]. Because $\sigma_p(\mathcal{A}) = \emptyset$, by [14, Propositions 9.1, 9.2] $\sigma(\mathcal{A}) = \sigma_c(\mathcal{A}) = \sigma_{ac}(\mathcal{A}) \cup \sigma_{sc}(\mathcal{A})$. It remains to show that $\sigma_{sc}(\mathcal{A}) = \emptyset$.

2.2 The resolvent of $\mathcal{A}$ and the proof $\sigma_{sc} = \emptyset$ To show that the spectrum is absolutely continuous, we will make use of [13, Theorem XIII.20]. According to this result, given $I = (a, b)$, $b > a$, to prove that $\sigma_{sc} \cap I = \emptyset$, it suffices to check that for all $v \in D$, with $\overline{D} = \mathcal{H}$,

$$\sup_{\epsilon \in (0,1)} \int\limits_I |\mathrm{Im}(R_{\mathcal{A}}(\lambda + i\epsilon)v, v)|^p \, d\lambda < \infty, \text{ for some } p > 1, \tag{2.8}$$

where $\mathcal{R}_{\mathcal{A}}(\lambda) = (\lambda \,\mathrm{Id} - \mathcal{A})^{-1}$.

A direct computation shows that the resolvent is defined by

$$\mathcal{R}_{\mathcal{A}}(\omega) = -\omega^{-1}\mathcal{S}_\omega \widetilde{\mathcal{S}}_\omega^T \mathcal{N}_\omega + \mathcal{K}_\omega, \quad \omega \in \mathbb{C} \setminus \mathbb{R},$$

where $\mathcal{N}_\omega$ is given by (1.16) and

$$\mathcal{K}_\omega = \begin{pmatrix} \omega^{-1} & 0 & 0 & 0 \\ 0 & (\omega\varepsilon(\omega))^{-1} & 0 & \frac{i\omega_p}{\omega^2\varepsilon(\omega)} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{i\omega_p}{\omega^2\varepsilon(\omega)} & 0 & (\omega\varepsilon(\omega))^{-1} \end{pmatrix}, \quad \mathcal{S}_\omega = \begin{pmatrix} -\partial_y, \\ \varepsilon^{-1}(\omega)\partial_x, \\ i\omega, \\ i\omega_p(\omega\varepsilon(\omega))^{-1}\partial_x \end{pmatrix},$$
$$\widetilde{\mathcal{S}}_\omega^T = \begin{pmatrix} -\partial_y, & \varepsilon^{-1}(\omega)\partial_x, & i\omega, & -i\omega_p(\omega\varepsilon(\omega))^{-1}\partial_x \end{pmatrix}.$$

Comparing the above expression with the explicit expression for $\mathcal{N}_\omega$, cf. (1.16), and the respective expression for the fundamental solution (1.12), we conclude that

LEMMA 2.5. *For all intervals I belonging to compact subsets of $\mathbb{R} \setminus \{0\} \cup \{\pm\omega_p\}$, and all $v \in C_0^\infty(\mathbb{R}^2)$, the inequality (2.8) holds true for any $p \geq 1$.*

Proof. Because the function $\omega \mapsto \varepsilon(\omega)$ is analytic and bounded on compact subsets of $\mathbb{C} \setminus \{\pm\omega_p, 0\}$, it suffices to prove that for all intervals as in the statement of the lemma, for all $v, \phi \in C_0^\infty$,

$$\sup_{\epsilon \in (0,1)} \int\limits_I |(\mathcal{N}_{\lambda+i\epsilon}v, \phi)|^p \, d\lambda < \infty \text{ for any } p \geq 1. \tag{2.9}$$

Case $I \subset (0, \omega_p)$. With the Plancherel identity we have

$$(\mathcal{N}_{\lambda+i\epsilon}v, \phi) = \int\limits_{\mathbb{R}^3} \mathcal{F}_x\mathcal{G}_{\lambda+i\epsilon}(\xi_x, y - y')\mathcal{F}_x v(\xi_x, y')\mathcal{F}_x\phi(\xi_x, y')d\xi_x dy' dy. \tag{2.10}$$

Recall that, according to (1.13),

$$\mathcal{F}_x\mathcal{G}_{\lambda+i\epsilon}(\xi_x,y)=\frac{\mathrm{e}^{i\kappa(\xi_x,\lambda+i\varepsilon)|y|}}{2i\sqrt{2\pi}\kappa(\xi_x,\lambda+i\epsilon)},\quad \kappa(\xi_x,\omega)=\sqrt{-\varepsilon^{-1}(\omega)\xi_x^2+\omega^2},\tag{2.11}$$

we remark the following: for all $\epsilon>0$,

- $\operatorname{Im}\kappa(\xi_x,\lambda+i\epsilon)\geq 0$, because $\operatorname{sign}\operatorname{Im}(-\varepsilon^{-1}(\lambda+i\epsilon))=-\operatorname{sign}\operatorname{Im}\frac{\omega_p^2}{\omega^2}>0$ and $\operatorname{sign}\operatorname{Im}(\lambda+i\epsilon)^2>0$.
- $\operatorname{Re}\kappa(\xi_x,\lambda+i\epsilon)>0$ by the properties of the square root (for $\operatorname{Im}\omega>0$, we have that $\operatorname{Im}\left(-\varepsilon^{-1}(\omega)\xi_x^2+\omega^2\right)>0$), and, moreover,

$$\left|\sqrt{-\varepsilon^{-1}(\lambda+i\epsilon)\xi_x^2+(\lambda+i\epsilon)^2}\right|=|\varepsilon(\lambda+i\epsilon)|^{-\frac{1}{2}}\left|\sqrt{-\xi_x^2+(\lambda+i\epsilon)^2\varepsilon(\lambda+i\epsilon)}\right|.\tag{2.12}$$

A direct computation yields

$$\begin{aligned}\left|\xi_x^2+(\lambda+i\epsilon)^2\varepsilon(\lambda+i\epsilon)\right|&=\left(\left(-\xi_x^2+\lambda^2-\epsilon^2-\omega_p^2\right)^2+4\epsilon^2\lambda^2\right)^{\frac{1}{2}},\\ |\varepsilon(\lambda+i\epsilon)|^{-1}&=\frac{\lambda^2+\epsilon^2}{\sqrt{(\lambda^2-\epsilon^2-\omega_p^2)^2+4\epsilon^2\lambda^2}}.\end{aligned}$$

Since $\lambda<\omega_p$, we have, for all $\epsilon>0$,

$$\begin{aligned}&\left(-\xi_x^2+\lambda^2-\epsilon^2-\omega_p^2\right)^2+4\epsilon^2\lambda^2-(\xi_x^2+\omega_p^2-\lambda^2)^2>0,\\ &(\lambda^2-\epsilon^2-\omega_p^2)^2+4\epsilon^2\lambda^2-(\lambda^2-\omega_p^2)^2>0,\end{aligned}$$

we conclude that

$$\begin{aligned}\left|-\xi_x^2+(\lambda+i\epsilon)^2\varepsilon(\lambda+i\epsilon)\right|&\geq\xi_x^2+\omega_p^2-\lambda^2,\\ |\varepsilon(\lambda+i\epsilon)|^{-\frac{1}{2}}&\geq\frac{\lambda}{\sqrt{\omega_p^2-\lambda^2}}.\end{aligned}$$

With these inequalities, (2.12) gives, for all $\epsilon>0$,

$$|\kappa(\xi_x,\lambda+i\epsilon)|=\left|\sqrt{-\varepsilon^{-1}(\lambda+i\epsilon)\xi_x^2+(\lambda+i\epsilon)^2}\right|\geq\lambda\sqrt{(\omega_p^2-\lambda^2)^{-1}\xi_x^2+1}.$$

Therefore, (2.11) can be bounded for all $0<\epsilon<1$ by

$$|\mathcal{F}_x\mathcal{G}_{\lambda+i\epsilon}(\xi_x,y)|\leq\lambda\sqrt{(\omega_p^2-\lambda^2)^{-1}\xi_x^2+1}\leq\lambda.$$

It is then easy to see that, cf. (2.10), for all $\xi_x\in\mathbb{R}$, $s',s>\frac{1}{2}$,

$$|\mathcal{F}_x\mathcal{G}_{\lambda+i\epsilon}(\xi_x,.)*\mathcal{F}_xv(\xi_x,.)|_{L^2_{-s'}(\mathbb{R})}\leq\lambda^{-1}\|\mathcal{F}_xv(\xi_x,.)\|_{L^2_s(\mathbb{R})},\forall\epsilon>0.$$

Let us come back to the expression (2.10), which we bound as

$$\begin{aligned}|\left(\mathcal{N}_{\lambda+i\epsilon}v,\phi\right)|&\lesssim\int_{\mathbb{R}}\|\mathcal{F}_x\left(\mathcal{N}_{\omega+i\epsilon}v\right)(\xi_x,.)\|_{L^2_{-s}(\mathbb{R})}\|\mathcal{F}_xv(\xi_x,.)\|_{L^2_s(\mathbb{R})}d\xi_x\\ &\lesssim\lambda^{-1}\int_{\mathbb{R}}\|\mathcal{F}_xv(\xi_x,.)\|_{L^2_s(\mathbb{R})}\|\mathcal{F}_x\phi(\xi_x,.)\|_{L^2_s(\mathbb{R})}d\xi_x\lesssim\lambda^{-1}\|v\|_{L^2_{s,\perp}}\|\phi\|_{L^2_{s,\perp}}.\end{aligned}$$

Because $\lambda\in L^p_{loc}(0,\omega_p)$, we conclude with the desired bound (2.9).
Case $I\subset(\omega_p,+\infty)$. Again, using an explicit expression for the integral kernel of the resolvent, namely (1.12), we remark the following:

$$\operatorname{Im}(\lambda+i\epsilon)\sqrt{\varepsilon(\lambda+i\epsilon)^2x^2+y^2}=\epsilon\operatorname{Re}\sqrt{(\lambda+i\epsilon)^2x^2+y^2}+\lambda\operatorname{Im}\sqrt{\varepsilon(\lambda+i\epsilon)^2x^2+y^2},$$

and because $\operatorname{Im}\varepsilon(\lambda+i\epsilon)^2>0$, for all $\epsilon>0$, we conclude that

$$\operatorname{Im}(\lambda+i\epsilon)\sqrt{\varepsilon(\lambda+i\epsilon)^2x^2+y^2}>0.$$

The function $z\mapsto H_0^{(1)}(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$, and, for all $|z|>R$, $\operatorname{Im} z\geq 0$, is uniformly bounded, cf. (1.14), by $O(R^{-\frac{1}{2}})$.

For $|z|\to 0$, recall (1.15). According to this expansion, combined with the fact that $\omega\mapsto\omega\sqrt{\varepsilon(\omega)x^2+y^2}$ is analytic in ω, $\operatorname{Im}\omega>0$, we have the bound

$$|H_0^{(1)}(z)|\lesssim\max(1,\log|z|),\quad |z|\leq 1.$$

By analyticity, the above two bounds yield the following bound valid for all z with $\operatorname{Im} z\geq 0$:

$$|H_0^{(1)}(z)|\lesssim\mathbb{1}_{|z|<\frac{1}{2}}\log|z|^{-1}+\mathbb{1}_{|z|\geq\frac{1}{2}}|z|^{-\frac{1}{2}}\leq\mathbb{1}_{|z|<\frac{1}{2}}\log|z|^{-1}+1. \tag{2.13}$$

Let us now bound the argument of the Bessel function in (FS). An explicit computation yields, for all $0<\epsilon<1$,

$$\begin{aligned}\left|\varepsilon(\lambda+i\epsilon)x^2+y^2\right|&\geq\left((\operatorname{Re}\varepsilon(\lambda+i\epsilon)x^2)^2+y^4+\operatorname{Im}\varepsilon(\lambda+i\epsilon)x^4\right)^{\frac{1}{2}}\\&=\left(\left(\frac{(\lambda^2-\epsilon^2-\omega_p^2)^2+4\epsilon^2\lambda^2}{(\lambda^2+\epsilon^2)^2}\right)^{\frac{1}{2}}x^4+y^4\right)^{\frac{1}{2}}\\&\geq\left(\frac{(\lambda^2-\omega_p^2)}{(\lambda^2+1)}x^4+y^4\right)^{\frac{1}{2}}\end{aligned}$$

where to obtain the last inequality we used the bound $(\lambda^2-\epsilon^2-\omega_p^2)^2+4\epsilon^2\lambda^2-(\lambda^2-\omega_p^2)^2>0$ valid for all $\epsilon>0$ and $\lambda>\omega_p$. It is then easy to conclude that

$$\left|\varepsilon(\lambda+i\epsilon)^2x^2+y^2\right|\geq\frac{1}{2}\sqrt{\lambda^2-\omega_p^2}|x|.$$

Therefore, (2.13) translates into: for all $\epsilon>0$, $\lambda\in I$,

$$\begin{aligned}\mathcal{G}_{\lambda+i\epsilon}(x,y)&\lesssim|\varepsilon(\lambda+i\epsilon)|^{\frac{1}{2}}\left(\log|x|^{-1}+\log\sqrt{\lambda^2-\omega_p^2}\right)\mathbb{1}_{|\mathbf{x}|\leq\frac{1}{2}}+1\\&\lesssim\log|x|^{-1}\mathbb{1}_{|\mathbf{x}|\leq\frac{1}{2}}+1.\end{aligned}$$

because $\varepsilon(\omega)$ is analytic when $(\operatorname{Re}\omega,\operatorname{Im}\omega)\in I\times(-\infty,+\infty)$, and $|\log\sqrt{\lambda^2-\omega_p^2}|\lesssim 1$ since $\omega_p\notin I$.

Finally, let us apply the above estimate to (2.9):

$$\begin{aligned}|(\mathcal{G}_{\lambda+i\epsilon}*v,\phi)|&\lesssim\int\limits_{\mathbb{R}^2}\phi(\mathbf{x})\left(\int\limits_{|\mathbf{x}-\mathbf{x}'|<\frac{1}{2}}\log|x-x'|v(\mathbf{x}')d\mathbf{x}'+\int\limits_{\mathbf{R}^2}v(\mathbf{x}')d\mathbf{x}'\right)d\mathbf{x}\\&\lesssim 1,\end{aligned}$$

and therefore the estimate (2.9) holds true with any p. ☐

We thus conclude that $\mathbb{R}\setminus\{0\}\cup\{\pm\omega_p\}\subset\sigma_{ac}(\mathcal{A})$. Because the support of the singularly continuous spectrum has no isolated points, and $\sigma_p(\mathcal{A})=\emptyset$, we conclude with

LEMMA 2.6. *$\sigma_{sc}(\mathcal{A})=\emptyset$, $\sigma_{ac}(\mathcal{A})=\mathbb{R}$.*

2.3 Proof of Theorem 2.1 By (2.3) and Lemma 2.3, we conclude that $\sigma(\mathcal{A})=\mathbb{R}$ and $\sigma_p(\mathcal{A})=\emptyset$. Lemma 2.6 yields the desired result.

3 Limiting absorption principle for the resolvent Because we are particularly interested in the behaviour of H_z, provided the respective right hand side data $fe^{i\omega t}$, for our needs it is sufficient to study $R_z(\lambda):=\mathbf{P}_z^T\mathcal{R}_A(\lambda)\mathbf{P}_z$, where $\mathbf{P}_z$ is the projection operator $\mathbf{P}_z=\mathbf{e}_3\mathbf{e}_3^T$. A straightforward computation yields

$$R_z(\omega)=\omega\mathcal{N}_\omega,\quad\omega\in\mathbb{C}\setminus\mathbb{R}.$$

The limiting absorption principle for the resolvent $R_z(\omega)$ reads.

THEOREM 3.1. *Let $\omega \in \mathbb{R}$. Then, given $s, s' > \frac{3}{2}$, for all $f \in L^2_s(\mathbb{R}^2)$, the following holds true :*

$$\lim_{\epsilon \to 0+} \| \left(R_z(\omega + i\epsilon) - R_z^+(\omega)\right) f\|_{L^2_{-s'}(\mathbb{R}^2)} = 0,$$

where

- $R_z^+(\omega) = \omega \mathcal{N}_\omega^+$, *if* $\omega \notin \{0, \pm\omega_p\}$, *with* $\mathcal{N}_\omega^+$ *defined in (1.16) for* $-\omega_p < \omega < \omega_p$, *and, when* $|\omega| > \omega_p$, $\mathcal{N}_\omega^+ \in \mathcal{B}(L^2_s, L^2_{-s'})$ *defined by*

$$\mathcal{N}_\omega^+ v = \mathcal{G}_\omega^+ * v, \quad \mathcal{G}_\omega^+(\mathbf{x}) = -\frac{i}{4}\sqrt{-\varepsilon(\omega)} H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2 + y^2}).$$

- $R_z^+(\omega) = 0$, *if* $\omega = \pm\omega_p$;
- $R_z^+(0)$ *is a continuous operator* $R_z^+(0) : L^2_{s,\perp}(\mathbb{R}^2) \to L^2_{-s',\perp}(\mathbb{R}^2)$, *defined for* $f \in L^2_{s,\perp}(\mathbb{R}^2)$ *as follows:*

$$(R_z^+(0)f)(x,y) = \frac{i\omega_p}{2\pi} \int_{\mathbb{R}^2} K_0(\omega_p|x - x'|) f(x', y') d\mathbf{x}'.$$

The proof of this result is subdivided into multiple sections. Evidently, it suffices to prove the result for $\omega > 0$.

3.1 Proof of Theorem 3.1 in the case when $0 < \omega < \omega_p$ The result follows straightforwardly from the same arguments as the proof of Theorem 4.1 in [8], see Appendix A.

3.2 Proof of Theorem 3.1 in the case when $\omega > \omega_p$ The result follows from the same arguments as in [2]. Because we need only an abridged version of the result, we repeat the arguments very briefly here. First of all we remark the following: in the topology of $L^1_{loc}(\mathbb{R}^2)$, $\mathcal{G}_{\omega+i\epsilon} \to \mathcal{G}_\omega^+$, where $\mathcal{G}_\omega^+$ is defined in the statement of the theorem. This can be proven based on the Lebesgue's dominated convergence theorem, by using $\operatorname{Im}\epsilon \geq 0$, the expansions (1.15), (1.14) and analyticity of $z \mapsto \mathcal{G}_z$. This shows that

$$\lim_{\epsilon \to 0+} (\mathcal{N}_{\omega+i\epsilon}\phi, \psi) = \left(\mathcal{N}_\omega^+\phi, \psi\right), \quad \forall \phi, \psi \in C_0^\infty(\mathbb{R}^2). \tag{3.1}$$

Boundedness of $\mathcal{N}_{\omega+i\epsilon}$ *in* $\mathcal{B}(L^2_s, L^2_{-s'})$. Moreover, we remark that, for all $v \in C_0^\infty(\mathbb{R}^2)$, $\omega > \omega_p$, $0 \leq \epsilon < 1$, we have

$$\begin{aligned}\|\mathcal{N}_{\omega+i\epsilon}v\|^2_{L^2_{-s'}} &= \int_{\mathbb{R}^2} (1 + |\mathbf{x}|^2)^{-s'} \left| \int_{\mathbb{R}^2} |\mathcal{G}_{\omega+i\epsilon}(\mathbf{x} - \mathbf{x}')v(\mathbf{x}')| \, d\mathbf{x}' \right|^2 d\mathbf{x} \\ &\lesssim \int_{\mathbb{R}^2} (1 + |\mathbf{x}|^2)^{-s'} \int_{\mathbb{R}} |\mathcal{G}_{\omega+i\epsilon}(\mathbf{x} - \mathbf{x}')|^2 (1 + |\mathbf{x}'|^2)^{-s} d\mathbf{x}' d\mathbf{x} \, \|v\|^2_{L^2_s}.\end{aligned}$$

Using (3) the above rewrites

$$\begin{aligned}\|\mathcal{N}_{\omega+i\epsilon}v\|^2_{L^2_{-s'}} \lesssim \|v\|^2_{L^2_s} \Bigg(&\int_{\mathbb{R}^2} (1 + |\mathbf{x}|^2)^{-s'} \int_{|\mathbf{x}-\mathbf{x}'| \leq \frac{1}{2}} \log^2 |\mathbf{x} - \mathbf{x}'| (1 + |\mathbf{x}'|^2)^{-s} d\mathbf{x}' d\mathbf{x} \\ &+ \int_{(\mathbf{x},\mathbf{x}') \in \mathbb{R}^4 :\, |\mathbf{x}-\mathbf{x}'| \geq \frac{1}{2}} (1 + |\mathbf{x}|^2)^{-s'} (1 + |\mathbf{x}'|^2)^{-s} d\mathbf{x}' d\mathbf{x} \Bigg).\end{aligned}$$

As $s, s' > \frac{3}{2}$, the two above integrals are obviously bounded.

Moreover, from (3) it follows that for all $0 \leq \varepsilon < 1$, we have the bound

$$\|\mathcal{N}_{\omega+i\epsilon}v\|^2_{L^2_{-s'}} \leq C_{s,s'} \|v\|^2_{L^2_s},$$

with $C_{s,s'}$ depending only on the interval I.

Therefore, we conclude that $\mathcal{N}_z, \mathcal{N}_\omega^+ \in \mathcal{B}(L^2_s, L^2_{-s'})$, $z \in \mathbb{C}^+$, $\operatorname{Re} z > \omega_p$.

Weak convergence of $R_z(\omega_n)f$ *in* $L^2_{-s'}$. Combining the above with (3.1) yields

$$R_z(\omega_n)f \rightharpoonup R(\omega)f \quad \text{in} \quad L^2_{-s'}(\mathbb{R}^2). \tag{3.2}$$

Boundedness of $\mathcal{N}_{\omega+i\epsilon}$ in $\mathcal{B}(L^2_s, H^2_{-s'})$. Let I be a bounded interval of (ω_p, ∞), and $0 \le \epsilon < 1$.

Let us show the bound $\|\mathcal{N}_{\omega+i\epsilon} f\|_{H^2_{-s'}} \lesssim \|\mathcal{N}_{\omega+i\epsilon} f\|_{L^2_{-s'}}$, valid for all ϵ sufficiently small. We will use the fact that $\|v\|_{H^2_{-s'}}$ is equivalent to $\|v\|_{L^2_{-s'}} + \|\Delta v\|_{L^2_{-s'}}$.

Because $f - (\omega+i\epsilon)^2 \mathcal{N}_{\omega+i\epsilon} f = \left(\varepsilon(\omega+i\epsilon)^{-1}\partial_x^2 + \partial_y^2\right)\mathcal{N}_{\omega+i\epsilon} f$, we remark that

$$\|\Delta \mathcal{N}_{\omega+i\epsilon} f\|^2_{L^2_{-s'}(\mathbb{R}^2)} \lesssim \left\| \mathcal{F}^{-1} \frac{\xi_x^2 + \xi_y^2}{\varepsilon(\omega+i\epsilon)^{-1}\xi_x^2 + \xi_y^2} \mathcal{F}(f - (\omega+i\epsilon)^2 \mathcal{N}_{\omega+i\epsilon} f) \right\|_{L^2_{-s'}}.$$

To prove that

$$\|\Delta \mathcal{N}_{\omega+i\epsilon} f\|^2_{L^2_{-s'}(\mathbb{R}^2)} \le C \|f\|_{L^2_s},$$

it suffices to show the bound

$$\left| \frac{\xi_x^2 + \xi_y^2}{(\varepsilon(\omega+i\epsilon)^{-1}\xi_x^2 + \xi_y^2)} \right| \le C. \tag{3.3}$$

This is however easy to see: $z \mapsto \varepsilon(z)^{-1}$ is continuous on $\mathbb{C} \setminus \{\pm\omega_p\}$, and $\operatorname{Re} \varepsilon(\omega)^{-1} > 0$ we conclude that $\operatorname{Re} \varepsilon(\omega+i\epsilon)^{-1} > c > 0$ for all sufficiently small ϵ. Then

$$\left|(\varepsilon(\omega+i\epsilon)^{-1}\xi_x^2 + \xi_y^2)\right| \ge \left(\operatorname{Re} \varepsilon(\omega+i\epsilon)^{-1}\xi_x^2 + \xi_y^2\right) \gtrsim |\boldsymbol{\xi}|^2.$$

The desired bound (3.3) then follows immediately.
Weak convergence of $R_z(\omega+i\epsilon)f$ in $H^2_{-s'}$. Because $R_z(\omega+i\epsilon)f$ is bounded in $H^2_{-s'}$ and because of the weak convergence (3.2), we conclude that $R_z(\omega+i\epsilon)f \rightharpoonup R_z(\omega)f$ in $H^2_{-s'}$.
Strong convergence of $R_z(\omega+i\epsilon)f$ in $H^2_{-s'}$. Because the results above are valid for any $s, s' > \frac{1}{2}$, we have in particular that $R_z(\omega+i\epsilon)f \rightharpoonup R_z(\omega)f$ in $H^2_{-s'+\delta}$ for all δ sufficiently small. Because $H^2_{-s'+\delta}$ is compactly embedded in $L^2_{-s'}$, we conclude that $R_z(\omega+i\epsilon)f \to R_z(\omega)f$ in $L^2_{-s'}$, hence the conclusion.

3.3 Proof of Theorem 3.1 in the case when $\omega = \omega_p$ Let us consider (1.12). We in particular see that

$$\mathcal{G}_{\omega_p+i\epsilon}(\mathbf{x}, \mathbf{y}) = -\frac{i}{4}\sqrt{\frac{2i\epsilon\omega_p - \varepsilon^2}{(\omega_p+i\epsilon)}} H_0^{(1)}((\omega_p+i\epsilon)\sqrt{\varepsilon(\omega_p+i\epsilon)x^2+y^2}).$$

Therefore, to prove that $\|u_{\omega_p+i\epsilon}\|_{L^2_{-s'}} \underset{\epsilon\to 0+}{\to} 0$, it is sufficient to control the $L^2_{-s'}$-norm of the following quantity:

$$v_\epsilon = H_0^{(1)}\left((\omega_p+i\epsilon)\sqrt{\varepsilon(\omega_p+i\epsilon)x^2+y^2}\right) * f \tag{3.4}$$

This is however quite easy to see. Using (2.13), we see that

$$\left|H_0^{(1)}\left((\omega_p+i\epsilon)\sqrt{\varepsilon(\omega_p+i\epsilon)x^2+y^2}\right)\right| \lesssim \left|\log|\varepsilon(\omega_p+i\epsilon)x^2+y^2|\right| \mathbb{1}_{|\varepsilon(\omega_p+i\epsilon)x^2+y^2|<1} + 1.$$

Because $|\varepsilon(\omega_p+i\epsilon)|\, x^2 \lesssim \sqrt{\epsilon}x^2$, we have the bound uniform in ϵ:

$$\left|H_0^{(1)}\left((\omega_p+i\epsilon)\sqrt{\varepsilon(\omega_p+i\epsilon)x^2+y^2}\right)\right| \lesssim |\log|y||\, \mathbb{1}_{|y|<1} + 1.$$

Using the above bound for (3.4) we have

$$\|v_\epsilon\|^2_{L^2_{-s'}} \lesssim \left(\int_{(\mathbf{x},\mathbf{x}')\in\mathbb{R}^2\times\mathbb{R}^2:\,|y-y'|<1} (1+|\mathbf{x}|^2)^{-s'} \log^2|y-y'|(1+|\mathbf{x}'|^2)^{-s} d\mathbf{x}d\mathbf{x}' + \int_{\mathbb{R}^2\times\mathbb{R}^2} (1+|\mathbf{x}|^2)^{-s'}(1+|\mathbf{x}'|^2)^{-s} d\mathbf{x}d\mathbf{x}' \right) \|f\|^2_{L^2_s}.$$

Both integrals are obviously bounded, hence, with $C > 0$ depending on s, s' but independent of ϵ, we have

$$\|v_\epsilon\|_{L^2_{-s'}} \leq C\|f\|_{L^2_s}$$

The above inequality and $\|u_{\omega_p+i\epsilon}\|_{L^2_{-s'}} \lesssim \sqrt{\epsilon}\|v_\epsilon\|_{L^2_{-s'}}$ prove the desired result.

3.4 Proof of Theorem 3.1 in the case when $\omega = 0$. Let us first start by showing that the resolvent $R_{i\epsilon} = i\epsilon\mathcal{N}_{i\epsilon}$ has a limit in $\mathcal{S}'$ as $\epsilon \to 0\pm$, $\epsilon \in \mathbb{R}$.

Let us express this limit in an explicit form. For this let us study the distributional limit of $\mathcal{G}_{\pm i\epsilon}$ as $\epsilon \to 0+$. Because $\mathcal{G}_{\pm i\epsilon} \in \mathcal{S}'(\mathbb{R}^2)$ for each ϵ, it suffices to study its partial Fourier transform, given by

$$\begin{aligned}\mathcal{F}\mathcal{G}_{\pm i\epsilon}(\boldsymbol{\xi}) &= -\frac{1}{2\pi(\xi_y^2 + \frac{\epsilon^2}{\epsilon^2+\omega_p^2}\xi_x^2 + \epsilon^2)} \\ &= -\frac{1}{2\pi}\left(\frac{1}{\xi_y - ir(\xi_x,\epsilon)} - \frac{1}{\xi_y + ir(\xi_x,\epsilon)}\right)\frac{1}{2ir(\xi_x,\epsilon)},\end{aligned}$$

with $r = \sqrt{\frac{\epsilon^2}{\epsilon^2+\omega_p^2}\xi_x^2 + \epsilon^2} = \epsilon\sqrt{\frac{1}{\epsilon^2+\omega_p^2}\xi_x^2 + 1}$. Using the same ideas as in the derivation of the Plemelj-Sokhotskii formula, cf. e.g. [16, pp. 75-76], one can show that

$$\lim_{\epsilon\to 0+}\frac{1}{\xi_y \pm ir(\xi_x,\epsilon)} = P.V.\frac{1}{\xi_y} \mp i\pi\delta_0(\xi_y) \text{ in } \mathcal{S}'.$$

Moreover, it holds pointwise (and in the L^1_{loc}-sense) $\lim\limits_{\epsilon\to 0+}\frac{i\epsilon}{2ir(\xi_x,\epsilon)} \to \frac{1}{\sqrt{\omega_p^{-2}\xi_x^2+1}}$. We thus have

$$\lim_{\epsilon\to 0+} \pm i\epsilon\mathcal{F}\mathcal{G}_{\pm i\epsilon}(\boldsymbol{\xi}) = \pm\frac{i}{2\sqrt{\xi_x^2+1}}\delta_{\xi_y}.$$

Let us thus introduce the following notation, for $f \in C_0^\infty(\mathbb{R}^2)$:

$$R_0^+ f := (\lim_{\epsilon\to 0+} i\epsilon\mathcal{G}_{i\epsilon}) * f, \tag{3.5}$$

so that, cf. [1, 9.6.21], with K_0 being the modified Bessel function:

$$R_0^+ f(\mathbf{x}) = \frac{i\omega_p}{2\pi}\int_{\mathbb{R}^2} K_0(\omega_p|x-x'|)f(x',y')d\mathbf{x}', \text{ and} \tag{3.6}$$

$$\mathcal{F}_x R_0^+ f(\xi_x, y) = \frac{1}{2i\sqrt{2\pi}\sqrt{\omega_p^{-2}\xi_x^2+1}}\int_{\mathbb{R}} \mathcal{F}_x f(\xi_x, y')dy'. \tag{3.7}$$

It remains to show that $R_z(\omega+i\epsilon) \to R_0^+$ in the strong topology of $\mathcal{B}(L^2_s, L^2_{-s'})$. A direct computation yields

$$\mathcal{F}_x(i\epsilon\mathcal{N}_{i\epsilon}f) - \mathcal{F}_x R_0^+ f = \frac{1}{2i\sqrt{2\pi}}\int_{\mathbb{R}}\left(\frac{e^{-\epsilon\sqrt{\frac{\xi_x^2}{\epsilon^2+\omega_p^2}+1}|y-y'|}}{\sqrt{\frac{\xi_x^2}{\epsilon^2+\omega_p^2}+1}} - \frac{1}{\sqrt{\omega_p^{-2}\xi_x^2+1}}\right)\mathcal{F}_x f(\xi_x, y')dy'.$$

The proof of the limiting absorption principle then follows like in the proof of Theorem 4.1 in [8], see Appendix A.

4 Hölder regularity of the resolvent for $\omega \in [-\omega_p, \omega_p]$ The goal of this section is to provide the Hölder estimates on the resolvent on the real line. First of all, a corollary of the proof of Theorem 3.1 reads.

Corollary 4.1. *For all $s, s' > \frac{3}{2}$, $f \in L^2_s(\mathbb{R}^2)$, the function $\omega \mapsto \mathcal{N}^+_\omega f$ is bounded in $L^2_{-s}(\mathbb{R}^2)$ on compact subsets of $\mathbb{R}\setminus\{0\}$.*

The principal result of this section reads.

THEOREM 4.2 (Hölder regularity of the resolvent). *The operator $R_z^+(\omega)$, $\omega \in \mathbb{R}$, defined in Theorem 3.1 satisfies the following regularity estimates in a strong operator topology.*

For all $s, s' > \frac{3}{2}$, $f \in L_s^2(\mathbb{R}^2)$,

$$R_z^+ f \in C^{0,1}((-\omega_p, \omega_p); L^2_{-s'}(\mathbb{R}^2)).$$

For all $\epsilon > 0$,

$$R_z^+ f \in C^{0,\frac{1}{4}}(B_\epsilon(\omega_p) \cup B_\epsilon(-\omega_p); L^2_{-s'}(\mathbb{R}^2).$$

Because the proofs of the above result are quite different for all the cases, we split the proof into several sections. Obviously, when $\omega \neq 0$, by Corollary 4.1, it suffices to show the result for $\mathcal{N}_\omega^+$. Also, we will prove the results for $\omega \geq 0$, the results for $\omega < 0$ being proven in the same way.

4.1 Proof of Theorem 4.2, case $0 < \omega < \omega_p$ Let $\omega \in (0, \omega_p)$ be fixed. We rewrite, for $h \in \mathbb{R}$, s.t. $\omega + h \in (0, \omega_p)$,

$$\mathcal{F}_x \mathcal{N}^+_{\omega+h} f = \frac{i}{2\sqrt{2\pi}} \int\limits_{\mathbb{R}} \frac{\mathrm{e}^{i\kappa(\xi_x, \omega+h)|y-y'|}}{\kappa(\xi_x, \omega+h)} \mathcal{F}_x f(\xi_x, y') dy' = \mathcal{F}_x \mathcal{N}_\omega^+ f + r(\omega),$$

and next use (A.8) to obtain

$$|r(\omega)(\xi_x, y)| \leq h \int\limits_{\mathbb{R}} |y - y'||f(\xi_x, y')| dy',$$

which, in turn results in

$$\|r(\omega)\|_{L^2_{-s',\perp}} \lesssim h \|f\|_{L^2_{s,\perp}}.$$

This proves that $\mathcal{N}_\omega^+ \in C^{0,1}((0, \omega_p), L^2_{-s',\perp}(\mathbb{R}))$.

4.2 Proof of Theorem 4.2, case $\omega = \omega_p$**.** Let $\epsilon > 0$ be fixed. We will prove the Hölder estimate by considering two cases: $B_\epsilon^-(\omega_p) := B_\epsilon(\omega_p) \cap \{\omega < \omega_p\}$ and $B_\epsilon^+(\omega_p) := B_\epsilon(\omega) \cap \{\omega > \omega_p\}$.

Hölder estimate in $B_\epsilon^-(\omega_p)$**.** Recall that $\mathcal{N}^+_{\omega_p} f = 0$, thus

$$\left(\mathcal{F}_x \mathcal{N}_\omega^+ f - \mathcal{F}_x \mathcal{N}^+_{\omega_p} f\right)(\xi_x, y) = \frac{1}{2i\sqrt{2\pi}} \int\limits_{\mathbb{R}} \frac{\mathrm{e}^{i\kappa(\xi_x, \omega)|y-y'|}}{\kappa(\xi_x, \omega)} \mathcal{F}_x f(\xi_x, y') dy'. \tag{4.1}$$

We rewrite

$$\kappa(\xi_x, \omega) = \sqrt{-\varepsilon(\omega)^{-1}\xi_x^2 + \omega^2} = (\omega_p^2 - \omega^2)^{-\frac{1}{2}} \omega \sqrt{\xi_x^2 + (\omega_p^2 - \omega^2)}.$$

Using the above expression, and bounding $|\mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|}| \leq 1$ in (4.1) yields

$$\left|\left(\mathcal{F}_x \mathcal{N}_\omega^+ f - \mathcal{F}_x \mathcal{N}^+_{\omega_p} f\right)(\xi_x, y)\right| \lesssim \frac{|\omega_p^2 - \omega^2|^{\frac{1}{2}}}{\left(\xi_x^2 + (\omega^2 - \omega_p^2)\right)^{\frac{1}{2}}} \int\limits_{\mathbb{R}} |\mathcal{F}_x f(\xi_x, y')| dy'.$$

Let us introduce the notation $\nu := |\omega_p^2 - \omega^2|^{\frac{1}{2}}$. To estimate the above in terms of ν, one would want to proceed classically, by using the Plancherel identity and the Cauchy-Schwarz inequality, which gives, for all $s', s > \frac{3}{2}$,

$$\left\|\left(\mathcal{F}_x \mathcal{N}_\omega^+ f - \mathcal{F}_x \mathcal{N}^+_{\omega_p} f\right)\right\|^2_{L^2_{-s',\perp}} \lesssim \int\limits_{\mathbb{R}} \frac{\nu^2}{\xi_x^2 + \nu^2} \|\mathcal{F}_x f(\xi_x, .)\|^2_{L^2_{-s,\perp}} d\xi_x. \tag{4.2}$$

However, we can see that for small ξ_x, the best possible estimate for $\frac{\nu^2}{\xi_x^2+\nu^2}$ is a constant, which would degenerate the Hölder estimate. The main idea then is to consider two separate cases: low-frequency case $|\xi_x| < \delta$, where $\delta < 1$, and the high-frequency case.

Step 1. An estimate for $|\xi_x| < \delta$**.** The main idea in deriving suitable for treating (4.2) for small frequencies lies in exploiting the fact that if $f \in L^2_s(\mathbb{R}^2)$, this implies that $\mathcal{F}_x f(., y') \in H^s(\mathbb{R})$ a.e. in $y' \in \mathbb{R}$, i.e., for s sufficiently large this allows to control $\frac{\nu^2}{\xi_x^2+\nu^2} \|\mathcal{F}_x f(\xi_x, .)\|^2_{L^2_{-s,\perp}}$ by using the Hölder regularity of $\mathcal{F}_x f(., y')$ in the vicinity of the origin.

When $|\xi_x| < \delta$, we will use the splitting

$$\mathcal{F}_x f(\xi_x, y') = \mathcal{F}_x f(\xi_x, y') - \mathcal{F}_x f(0, y') + \mathcal{F}_x f(\xi_x, y').$$

Without loss of generality, let us assume that $\xi_x > 0$.

By the assumption, $f \in L^2_s(\mathbb{R}^2)$, for $s > \frac{3}{2}$. This implies that $\mathcal{F}_x f \in H^{s,0}(\mathbb{R}^2)$, with $s > \frac{3}{2}$. Therefore, a.e. in y', $\mathcal{F}_x f(.,y) \in H^s(\mathbb{R})$, and thus we have the bound:

$$\begin{aligned}
|\mathcal{F}_x f(\xi_x, y') - \mathcal{F}_x f(0, y')| &\lesssim \xi_x \left(\int_0^{\xi_x} |\partial_{\xi_x} \mathcal{F}_x f(\xi_x', y')|^2 d\xi_x' \right)^{\frac{1}{2}} \\
&\lesssim \xi_x \left(\int_{-\infty}^{\infty} |\partial_{\xi_x} \mathcal{F}_x f(\xi_x', y')|^2 d\xi_x' \right)^{\frac{1}{2}} \\
&\leq \xi_x \|f(., y')\|_{L^2_{\frac{1}{2}}},
\end{aligned}$$

where the last inequality follows from the Plancherel theorem. On the other hand, for all $s > \frac{1}{2}$,

$$|\mathcal{F}_x f(0, y')| \lesssim |\int_{\mathbb{R}} f(x, y') dx| \lesssim \|f(., y')\|_{L^2_s}.$$

Thus (4.2) for $|\xi_x| < \delta$ rewrites

$$\begin{aligned}
\left\| \left(\mathcal{F}_x \mathcal{N}^+_\omega f - \mathcal{F}_x \mathcal{N}^+_{\omega_p} f \right)(\xi_x, .) \right\|_{L^2_{-s'}(\mathbb{R})} &\lesssim \frac{\nu \xi_x}{\sqrt{\xi_x^2 + \nu^2}} \|f\|_{L^2_{\frac{1}{2}, s}} + \frac{\nu}{\sqrt{\xi_x^2 + \nu^2}} \|f\|_{L^2_s} \\
&\lesssim \frac{\nu}{\sqrt{\xi_x^2 + \nu^2}} \|f\|_{L^2_s},
\end{aligned} \tag{4.3}$$

where in the last inequality we used $|\xi_x| < \delta < 1$.

Step 2. An estimate for $|\xi_x| > \delta$. In this case we use (4.2), which implies that for all $s, s' > \frac{1}{2}$, we have

$$\left\| \left(\mathcal{F}_x \mathcal{N}^+_\omega f - \mathcal{F}_x \mathcal{N}^+_{\omega_p} f \right)(\xi_x, .) \right\|_{L^2_{-s'}(\mathbb{R})} \lesssim \frac{\nu}{\sqrt{\delta^2 + \nu^2}} \|\mathcal{F}_x f(\xi_x, .)\|_{L^2_s(\mathbb{R})}. \tag{4.4}$$

Step 3. Hölder regularity bound in $B^-_\epsilon(\omega_p)$. Combining (4.3) for $|\xi_x| < \delta$ and (4.4) for $|\xi_x| > \delta$ into (4.2) yields the following bound:

$$\begin{aligned}
\int_{-\infty}^{\infty} \left\| \left(\mathcal{F}_x \mathcal{N}^+_\omega f - \mathcal{F}_x \mathcal{N}^+_{\omega_p} f \right)(\xi_x, .) \right\|^2_{L^2_{-s'}(\mathbb{R})} d\xi_x &\lesssim \|f\|^2_{L^2_s} \int_{-\delta}^{\delta} \frac{\nu^2}{\xi_x^2 + \nu^2} d\xi_x \\
&\quad + \frac{\nu^2}{\delta^2 + \nu^2} \|f\|^2_{L^2_{s,\perp}}.
\end{aligned}$$

Evaluating the integral above and bounding $\|f\|^2_{L^2_{s,\perp}} \leq \|f\|^2_{L^2_s}$, we obtain

$$\int_{-\infty}^{\infty} \left\| \left(\mathcal{F}_x \mathcal{N}^+_\omega f - \mathcal{F}_x \mathcal{N}^+_{\omega_p} f \right)(\xi_x, .) \right\|^2_{L^2_{-s'}(\mathbb{R})} d\xi_x \lesssim \left(\nu \operatorname{atan} \frac{\delta}{\nu} + \frac{\nu^2}{\delta^2 + \nu^2} \right) \|f\|^2_{L^2_s}.$$

Therefore, it suffices to take $\delta < 1$ a fixed constant, which would result in the following bound:

$$\left\| \mathcal{N}^+_\omega f - \mathcal{N}^+_{\omega_p} f \right\|^2_{L^2_{-s',\perp}} \lesssim \nu \|f\|^2_{L^2_s}, \quad \nu = \sqrt{|\omega^2 - \omega_p^2|}.$$

As $\nu = O(\sqrt{\omega_p - \omega})$, this shows that $\mathcal{N}^+_\omega \in C^{0,\frac{1}{4}}((\omega_p - \epsilon, \omega_p])$.

Hölder estimate in $B^+_\epsilon(\omega_p)$. In this case

$$\mathcal{N}^+_\omega f - \mathcal{N}^+_{\omega_p} f = \frac{-i\sqrt{\varepsilon(\omega)}}{4} g_\omega, \tag{4.5}$$

where

$$g_\omega(\mathbf{x}) = \int\limits_{\mathbb{R}^2} H_0^{(1)}\left(\omega\sqrt{\varepsilon(\omega)(x-x')^2+(y-y')^2}\right) f(\mathbf{x}')d\mathbf{x}'.$$

We remark that g_ω satisfies

$$\|g_\omega\|_{L^2_{-s'}} \lesssim \|f\|_{L^2_s}, \text{ for all } \omega \in B_\epsilon^+(\omega_p). \tag{4.6}$$

The above bound follows from the arguments of the proof of Theorem 3.1. With (4.5) we conclude that

$$\left\|\mathcal{N}_\omega^+ f - \mathcal{N}_{\omega_p}^+ f\right\|^2_{L^2_{-s',\perp}} \lesssim \nu\|f\|^2_{L^2_s}, \quad \nu = \sqrt{|\omega^2-\omega_p^2|}. \tag{4.7}$$

4.3 Proof of Theorem 4.2, case $\omega = 0$ Let $\omega \in B_\epsilon(0)$, $\epsilon > 0$. Using (3.7) we have

$$\mathcal{F}_x\left(R_z^+(\omega) - R_z^+(0)\right) f(\xi_x, y) = \frac{1}{2i\sqrt{2\pi}} \int\limits_{\mathbb{R}} d_{\xi_x,\omega}(y-y')\mathcal{F}_x f(\xi_x, y')dy', \tag{4.8}$$

where, with $\widetilde{\kappa}(\xi_x,\omega) = \sqrt{(\omega_p^2-\omega^2)^{-1}\xi_x^2+1}$, we defined

$$\begin{aligned}
d_{\xi_x,\omega}(y-y') &= \frac{\omega e^{i\kappa(\xi_x,\omega)|y-y'|}}{\kappa(\xi_x,\omega)} - \frac{1}{\sqrt{\omega_p^{-2}\xi_x^2+1}} = \frac{e^{i\omega\widetilde{\kappa}(\xi_x,\omega)|y-y'|}}{\widetilde{\kappa}(\xi_x,\omega)} - \frac{1}{\widetilde{\kappa}(\xi_x,0)} \\
&= \widetilde{\kappa}^{-1}(\xi_x,0)\left(e^{i\omega\widetilde{\kappa}(\xi_x,\omega)|y-y'|}-1\right) \qquad (4.9)\\
&\quad + e^{i\omega\widetilde{\kappa}(\xi_x,\omega)|y-y'|}\left(\widetilde{\kappa}^{-1}(\xi_x,\omega) - \widetilde{\kappa}^{-1}(\xi_x,0)\right). \qquad (4.10)
\end{aligned}$$

It remains to remark that

$$\left|\widetilde{\kappa}^{-1}(\xi_x,0)\right| \le C_\epsilon \frac{1}{\sqrt{\xi_x^2+1}}, \quad \left|e^{i\omega\widetilde{\kappa}(\xi_x,\omega)|y-y'|}\right| \le 1,$$

and

$$\begin{aligned}
&\left|e^{i\omega\widetilde{\kappa}(\xi_x,\omega)|y-y'|}-1\right| \lesssim \min(1, \omega\widetilde{\kappa}(\xi_x,\omega)|y-y'|) \lesssim |\omega|\max(|\xi_x|,1)|y-y'|,\\
&\left|\widetilde{\kappa}^{-1}(\xi_x,\omega) - \widetilde{\kappa}(\xi_x,0)\right| \lesssim |\omega|\frac{|\xi_x|}{(\xi_x^2+1)^{\frac{3}{2}}}.
\end{aligned}$$

Therefore, we can bound (4.9) by

$$|d_{\xi_x,\omega}(y-y')| \lesssim |\omega|\max(|y-y'|,1),$$

which, together with (4.8), by using the same argument as in the proof of the case $0<\omega<\omega_p$, shows that $R_z^+ \in C^{0,1}(B_\epsilon(0))$.

5 Proof of the limiting amplitude principle (Theorem 1.5) While this proof is quite classical, we nonetheless repeat its main points here (since in the original work of Eidus [9] it is the second order problems that are considered, and in the thesis [5] the respective results for the first order system are available only in the weak form).

Step 1. A computable expression for the solution of (2.2). By the Stone theorem (cf. [14, Proposition 6.1] as well as [11, Section 4.2]), provided that $\mathcal{F} \in L^1(0,T;\mathcal{H})$, the unique solution to the problem (2.2) is given by

$$\mathcal{U}(t) = \int\limits_0^t e^{i\mathcal{A}(t-\tau)}\mathcal{F}(\tau)d\tau.$$

Using the spectral theorem and the functional calculus [14, Theorem 5.7, Sectoin 5.3], we can rewrite the above integral as follows:

$$\mathcal{U}(t) = \int\limits_0^t \int\limits_{\mathbb{R}} e^{i\lambda(t-\tau)}dE_{\mathcal{A}}\mathcal{F}(\tau)d\tau,$$

where $E_{\mathcal{A}}$ is the spectral measure associated to the operator $\mathcal{A}$.

Since we are interested in the behaviour of H_z only, in the case when $\mathcal{F} = \left(0, 0, -if\mathrm{e}^{i\omega t}, 0\right)^T$, we rewrite the above in a more suitable form, introducing $\mathbf{p}_z = (0, 0, 1, 0)^T$:

$$H_z(t) = -i\mathbf{p}_z^T \int_0^t \int_{\mathbb{R}} \mathrm{e}^{i\lambda(t-\tau)} dE_{\mathcal{A}} \left(\mathbf{p}_z f \mathrm{e}^{i\omega\tau}\right) d\tau. \tag{5.1}$$

Let us denote $E_z := \mathbf{p}_z^T E_{\mathcal{A}} \mathbf{p}_z$; by construction, this is the spectral measure of the self-adjoint (on the space $L^2(\mathbb{R}^2)$) operator $\mathcal{A}_z = \mathbf{p}_z^T \mathcal{A} \mathbf{p}_z$. We then obviously have

$$H_z(t) = -i \int_0^t \int_{\mathbb{R}} \mathrm{e}^{i\lambda(t-\tau)} \mathrm{e}^{i\omega\tau} dE_z f d\tau. \tag{5.2}$$

For $v, q \in L^2(\mathbb{R}^2)$, we denote by $\mu_{v,q}(M) = (E_z(M)v, q)$. We then have, by the Fubini theorem

$$\left(\int_0^t \int_{\mathbb{R}} \mathrm{e}^{i\lambda(t-\tau)} \mathrm{e}^{i\omega\tau} dE_z v d\tau, q\right) = \int_0^t \int_{\mathbb{R}} \mathrm{e}^{-i\lambda(t-\tau)} \mathrm{e}^{-i\omega\tau} d\mu_{v,q}(\lambda) = i^{-1} \int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda} d\mu_{v,q}(\lambda). \tag{5.3}$$

Since $F(\lambda) = \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda}$ is bounded on $\mathbb{R}$, the function $\int_{\mathbb{R}} F(\lambda) dE_z(\lambda)$ defines a bounded operator on $L^2(\mathbb{R}^2)$, cf. [14, p.92]. Therefore,

$$\int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda} d\mu_{v,q}(\lambda) = \left(\int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda} dE_A v, q\right), \quad \forall v, q \in L^2(\mathbb{R}^2).$$

Comparing the above to (5.3) we obtain

$$\int_0^t \int_{\mathbb{R}} \mathrm{e}^{i\lambda(t-\tau)} \mathrm{e}^{i\omega\tau} dE_z d\tau = i^{-1} \int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda} dE_z. \tag{5.4}$$

Rewriting the spectral measure. The Stone's formula [14, Proposition 5.14] yields an explicit expression for the spectral measure, $-\infty < a < b < \infty$,

$$\frac{1}{2}\left(E_A(a, b) + E_A([a, b])\right) = \frac{1}{2\pi i} \lim_{\nu \to 0+} \int_a^b \left(R(\lambda + i\nu) - R(\lambda - i\nu)\right) d\lambda \tag{5.5}$$

In the above the integral is understood as a Riemann operator integral in the uniform operator topology $\mathcal{B}(\mathcal{H}, \mathcal{H})$, and the limit as a strong limit in $\mathcal{H}$.

For E_z, the Stone's formula rewrites

$$\frac{1}{2}\left(E_z(a, b) + E_z([a, b])\right) = \frac{1}{2\pi i} \lim_{\nu \to 0+} \int_a^b \left(R_z(\lambda + i\nu) - R_z(\lambda - i\nu)\right) d\lambda. \tag{5.6}$$

Recall that $f \in L^2_s$, for $s > \frac{3}{2}$, so that, by Theorem 3.1, $\lim\limits_{\nu \to 0+} R_z(\lambda + i\nu) f = R_z^+ f$ in $L^2_{-s'}$, $s' > \frac{3}{2}$. Correspondingly, $\lim\limits_{\nu \to 0+} R_z(\lambda - i\nu) f = R_z^- f$ in $L^2_{-s'}$.

Let $I_{L^2 \hookrightarrow L^2_{-s'}}$ be the embedding operator of L^2 into $L^2_{-s'}$. Then (5.5) can be rewritten as follows:

$$\frac{1}{2} \mathcal{I}_{L^2 \hookrightarrow L^2_{-s'}} \left(E_z(a, b) + E_z([a, b])\right) f = \frac{1}{2\pi i} \int_a^b \left(R^+(\lambda) - R^-(\lambda)\right) f d\lambda,$$

where the integral can be understood as the Bochner integral (since $\| \left(R^+(.) - R^-(.)\right) f \|_{L^2_{-s'}} \in L^1_{loc}(a, b)$). To obtain the above from (5.6) we used the Lebesgue's dominated convergence theorem for Bochner integrals.

Let us define

$$\theta(\lambda) := R^+(\lambda) - R^-(\lambda).$$

By Theorem 4.2, for all $f \in L^2_s$, $\theta(.)f$ is continuous in $L^2_{-s'}$. Using the results of [9, Lemma 1.3], we deduce that

$$\mathcal{I}_{L^2 \hookrightarrow L^2_{-s'}} \int_a^b \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda} dE_z f = \int_a^b \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda} \theta(\lambda) f d\lambda,$$

where the integral in the right hand side is the Bochner integral in $L^2_{-s'}$. Because the integral in the left-hand side is well-defined, by the spectral theory, for $a = -\infty$, $b = +\infty$, we also have

$$\mathcal{I}_{L^2 \hookrightarrow L^2_{-s'}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\lambda - \omega} dE_z f = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\lambda - \omega} \theta(\lambda) f d\lambda,$$

where the integral in the right-hand side is understood as a strong $L^2_{-s'}$-limit as $a \to -\infty$, $b \to +\infty$ of the Bochner integrals over the intervals (a, b).

We thus obtain the following expression for H_z from (5.1) and (5.4), where we will omit the injection operator remembering that the integrals are defined in the $L^2_{-s'}$-strong topology:

$$H_z(t) = -\int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t} - \mathrm{e}^{i\lambda t}}{\omega - \lambda} \theta(\lambda) f d\lambda. \tag{5.7}$$

In what follows, all the limits will be understood as limits in a strong $L^2_{-s'}$-topology (and so the Bochner integrals).

Step 2. Splitting of the integral in (5.7). We then rewrite the above as (we will justify further that the principal values below are well-defined):

$$\begin{aligned} H_z(t) &= P.V. \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\lambda t}}{\omega - \lambda} \theta(\lambda) d\lambda - P.V \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega t}}{\omega - \lambda} \theta(\lambda) f d\lambda \\ &= \mathrm{e}^{i\omega t} \left(\mathcal{I}_1 - \mathcal{I}_2 \right), \end{aligned} \tag{5.8}$$

$$\mathcal{I}_1 = -P.V. \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i(\lambda - \omega)t}}{\omega - \lambda} \theta(\lambda) f d\lambda = P.V. \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\lambda t}}{\lambda} \theta(\lambda + \omega) f d\lambda, \tag{5.9}$$

$$\mathcal{I}_2 = P.V. \int_{-\infty}^{\infty} \frac{1}{\omega - \lambda} \theta(\lambda) f d\lambda.$$

Step 3.1. Rewriting $\mathcal{I}_1$. We rewrite $\mathcal{I}_1$ as follows (with the index 'p' standing for 'principal' and 'r' standing for 'remainder'), with W sufficiently large,

$$\mathcal{I}_1 = \mathcal{I}_1^p + \mathcal{I}_1^r, \quad \mathcal{I}_1^p(t) = P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{\lambda} \theta(\lambda + \omega) f d\lambda, \quad \mathcal{I}_1^r(t) = \int_{|\lambda| > W} \frac{\mathrm{e}^{i\lambda t}}{\lambda} \theta(\lambda + \omega) f d\lambda.$$

Step 3.1.1. Controlling $\mathcal{I}_1^p$. We rewrite

$$\mathcal{I}_1^p(t) = P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{\lambda} \left(\theta(\lambda + \omega) - \theta(\omega) \right) f d\lambda + P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{\lambda} \theta(\omega) f d\lambda.$$

Step 3.1.1.a) Controlling $P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{i\lambda} \theta(\omega) f d\lambda$. By the residue theorem, we have

$$P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{\lambda} d\lambda = \lim_{\varepsilon \to 0+} \left(\int_{\mathcal{C}_\varepsilon} \frac{\mathrm{e}^{i\lambda t}}{\lambda} d\lambda - \int_{\mathcal{C}_W} \frac{\mathrm{e}^{i\lambda t}}{\lambda} d\lambda \right),$$

where $\mathcal{C}_R$ is a boundary of a circle $B_R(0)$ lying in $\mathbb{C}^+$ and oriented directly. A classical computation yields

$$P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{\lambda} d\lambda = i\pi + O((Wt)^{-1}). \tag{5.10}$$

Step 3.1.1.b) Controlling $P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{\lambda}(\theta(\lambda+\omega)-\theta(\omega))f d\lambda$. Since

$$\frac{1}{\lambda}\left(\theta(\lambda+\omega)-\theta(\omega)\right) \in L^1((-W,W);L^2_{-s'}),$$

we have by the Riemann-Lebesgue lemma for Bochner integrals [3, Theorem 1.8.1]:

$$P.V. \int_{-W}^{W} \frac{\mathrm{e}^{i\lambda t}}{\lambda}(\theta(\lambda+\omega)-\theta(\omega)) f d\lambda = o_W(1), \quad t \to +\infty,$$

where by o_W we indicate that the behaviour of the above quantity as $t \to +\infty$ depends on W.

Step 3.1.1.c) Conclusion. Therefore, for all $W > 0$, as $t \to +\infty$,

$$\mathcal{I}_1^p(t) = i\pi\theta(\omega)f + O((Wt)^{-1}) + o_W(1). \tag{5.11}$$

Step 3.1.2. Controlling $\mathcal{I}_1^r$. We rewrite $\mathcal{I}_1^r$ as follows, cf. (5.9),

$$\mathcal{I}_1^r(t) = \int_{|\omega-\lambda|>W} \frac{\mathrm{e}^{i(\omega-\lambda)t}}{\lambda-\omega} dE_A f.$$

By the properties of the spectral integrals [14, Proposition 4.15], we have

$$\begin{aligned} \left\| \int_{|\omega-\lambda|>W} \frac{\mathrm{e}^{i(\omega-\lambda)t}}{\lambda-\omega} dE_A f \right\|_{L^2}^2 &= \int_{|\omega-\lambda|>W} \left| \frac{\mathrm{e}^{-i(\omega-\lambda)t}}{(\omega-\lambda)} \right|^2 (dE_A f, f) \\ &\lesssim W^{-2}\|f\|_{L^2}^2. \end{aligned} \tag{5.12}$$

Step 3.1.3. Conclusion about $\mathcal{I}_1$. Combining (5.11) and (5.12), we obtain the following estimate, for any $W > 0$:

$$\mathcal{I}_1(t) = i\pi\theta(\omega)f + O((Wt)^{-1}) + o_W(1) + O(W^{-1}), \qquad \text{as } t \to +\infty.$$

Because W in the above can be chosen arbitrarily, we have

$$\lim_{t\to+\infty} \mathcal{I}_1(t) = i\pi\theta(\omega)f.$$

Step 3.2. Controlling $\mathcal{I}_2$. Recall that

$$\mathcal{I}_2 = P.V. \int_{-\infty}^{\infty} \frac{1}{\omega-\lambda}\theta(\lambda) f d\lambda. \tag{5.13}$$

To find a suitable representation for $\mathcal{I}_2$ we need a Plemelj-Sokhotski type lemma for Bochner integrals. In particular, a straightforward adaptation of the proof of the Plemelj-Sokhotski formula in [16, pp. 75-76] to Bochner integrals yields the following, for all $a > 0$:

$$\lim_{\nu\to 0+} \int_{-a}^{a} \frac{\theta(\lambda)f}{\omega+i\nu-\lambda} d\lambda = P.V. \int_{-a}^{a} \frac{\theta(\lambda)f}{\omega-\lambda} d\lambda - i\pi\theta(\omega)f, \tag{5.14}$$

where all the quantities are well-defined because $\theta(\lambda)f \in C^{0,\alpha}(\mathbb{R}; L^2_{-s'}(\mathbb{R}^2))$, which is a direct corollary of Theorem 4.2.

To pass to the limit as $a \to +\infty$ in the above, we recall that, with Theorem 3.1, we have:

$$\lim_{\nu \to 0+} \int_{-\infty}^{\infty} \frac{\theta(\lambda) f}{\omega + i\nu - \lambda} f d\lambda \equiv \lim_{\nu \to 0+} \mathcal{R}_z(\omega + i\nu) f = \mathcal{R}_z^+ f.$$

Thus, (5.14) rewrites

$$\mathcal{R}_z^+ f - \lim_{\nu \to 0+} \int_{|\lambda|>a} \frac{\theta(\lambda) f}{\omega - \lambda - i\nu} d\lambda = P.V. \int_{-a}^{a} \frac{\theta(\lambda) f}{\omega - \lambda} d\lambda - i\pi\theta(\omega) f. \tag{5.15}$$

Because, by [14, Proposition 4.15], the following bound holds uniformly for all $\nu \geq 0$:

$$\| \int_{|\lambda|>a} \frac{\theta(\lambda) f}{\omega + i\nu - \lambda} d\lambda \|_{L^2}^2 \lesssim a^{-2} \|f\|_{L^2},$$

passing to the limit in (5.14) as $a \to +\infty$ yields

$$\mathcal{R}_z^+ f = P.V. \int_{-\infty}^{\infty} \frac{\theta(\lambda) f}{\omega - \lambda} d\lambda - i\pi\theta(\omega) f. \tag{5.16}$$

Comparing the above to (5.13), we conclude that

$$\mathcal{I}_2 = \mathcal{R}_z^+ f + i\pi\theta(\omega) f. \tag{5.17}$$

Step 4. Summary. Combining (5.8), (5), (5.17), we obtain the desired estimate, as $t \to +\infty$:

$$\lim_{t \to +\infty} \|H_z(t) + \mathrm{e}^{i\omega t} \mathcal{R}_z^+ f\|_{L^2_{-s'}} = 0.$$

The result follows from the explicit expression for $\mathcal{R}_z^+$, given in Theorem 3.1.

6 Conclusion We have proven in this report the limiting amplitude principle for wave propagation in strongly magnetized cold plasma in free space.

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Appendix A. Proof of Theorem 1.4. First of all, let us remark the following. For $\omega \in \mathbb{R}$, from the definition of $\kappa(\xi_x, \omega)$ (1.13), it follows that

$$\frac{1}{2}\left((-\varepsilon(\omega))^{-1/2}|\xi_x| + \omega\right) \leq \kappa(\xi_x, \omega) = \sqrt{-\varepsilon(\omega)\xi_x^2 + \omega^2} \leq (-\varepsilon(\omega))^{-1/2}|\xi_x| + \omega.$$

Therefore, by (1.8), (1.9), an equivalent norm in $H^1_{p,\perp}$ is given by

$$\|v\|^2_{H^1_{p,\perp}} \sim \|\kappa(\xi_x, \omega)\mathcal{F}_x v\|^2_{L^2_{p,\perp}} + \|\partial_y \mathcal{F}_x v\|^2_{L^2_{p,\perp}}. \tag{A.1}$$

The constants in norm-equivalence inequalities depend on ω only.

Proof. The proof is quite easy and is based on the explicit representation of the operator $\mathcal{N}_\omega$. Let us fix $s, s' > \frac{3}{2}$. Let us set $r_n := \mathcal{N}_{\omega_n} f - \mathcal{N}^+_\omega f$, $\kappa_n := \sqrt{-\varepsilon^{-1}(\omega_n)\xi_x^2 + \omega_n^2}$. Using (1.13), we obtain

$$\kappa \mathcal{F}_x r_n(\xi_x, y) = \frac{1}{2i\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{\kappa}{\kappa_n} \mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right) \mathcal{F}_x f(\xi_x, y') dy', \tag{A.2}$$

$$\partial_y \mathcal{F}_x r_n(\xi_x, y) = \frac{1}{2i\sqrt{2\pi}} \int_{\mathbb{R}} \left(\mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right) \mathcal{F}_x f(\xi_x, y') dy'. \tag{A.3}$$

Recall the norm equivalence (A.1). We will show that $\lim\limits_{n\to+\infty} \|\kappa \mathcal{F}_x r_n\|_{L^2_{-s',\perp}} = 0$; the analogous result for $\partial_y \mathcal{F}_x r_n$ will follow in the same way.

Step 1. A few auxiliary bounds. First, remark that, as $\operatorname{Im} \kappa_n \geq 0$,

$$\left|\frac{\kappa}{\kappa_n} \mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right| \lesssim \left|\frac{\kappa}{\kappa_n} - 1\right| + \left|\mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right|. \tag{A.4}$$

Evidently, we have in particular

$$\left|\frac{\kappa}{\kappa_n} \mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right| \lesssim 1. \tag{A.5}$$

A finer bound can be obtained by remarking that the function

$$\omega \to \kappa(\omega) := \sqrt{\omega^2 - \varepsilon^{-1}(\omega)\xi_x^2}$$

is uniformly Lipschitz on all compact subsets of $\{z : 0 < \operatorname{Re} z < \omega_p\}$. Let $\delta > 0$ be sufficiently small. With $B^+_\delta(\omega) = \mathbb{C}^+ \cap B_\delta(\omega)$,

$$|\kappa - \kappa_n| \lesssim \sup_{z \in B^+_\delta(\omega)} \left|\frac{\partial \kappa}{\partial \omega}(z)\right| |\omega - \omega_n|, \quad \left|\frac{\partial \kappa}{\partial \omega}(z)\right| = \left|\frac{2z - (\varepsilon^{-1}(z))'\xi_x^2}{2\sqrt{z^2 - \varepsilon^{-1}(z)\xi_x^2}}\right|.$$

Therefore,

$$|\kappa - \kappa_n| \lesssim \max(|\xi_x|, 1)|\omega_n - \omega|. \tag{A.6}$$

Similarly, since for $|\omega_n - \omega| \to 0$, $|\kappa_n| \gtrsim |\xi_x| + 1$, we conclude from the above that

$$\left|\frac{\kappa}{\kappa_n} - 1\right| \lesssim |\omega_n - \omega|. \tag{A.7}$$

As for the second term in (A.4), since $\operatorname{Im} \kappa_n > 0$, the same argument as above gives

$$\left|\mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right| \lesssim |y - y'||\kappa_n - \kappa| \overset{(A.6)}{\lesssim} |\omega_n - \omega||y - y'| \max(|\xi_x|, 1). \tag{A.8}$$

Combining (A.7) and (A.8), and using the fact that all the quantities in the left-hand-side of (A.4) are bounded uniformly in y, ξ_x and for all ω_n sufficiently close to ω, we obtain the following bound valid for all n sufficiently large:

$$\left|\frac{\kappa_n}{\kappa} \mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right| \lesssim \min(1, |\omega_n - \omega|)|y - y'| \max(|\xi_x|, 1). \tag{A.9}$$

Step 2. Splitting in high and low frequencies. Next, let us split

$$\begin{aligned}&\mathcal{F}_x r_n(\xi_x,y)=\hat{r}_n^{lf}(\xi_x,y)+\hat{r}_n^{hf}(\xi_x,y),\\&\hat{r}_n^{lf}(\xi_x,y)=\mathbb{1}_{|\xi_x|<A}\hat{r}_n(\xi_x,y),\quad \hat{r}_n^{hf}(\xi_x,y)=\mathbb{1}_{|\xi_x|\geq A}\hat{r}_n(\xi_x,y),\end{aligned}$$

where $A>1$ will be chosen later. We will estimate these two quantities separately.
Step 2.1. Estimating $\hat{r}_n^{hf}(\xi_x,y)$. We use a uniform bound (A.5) in (A.2), which yields

$$|\kappa\hat{r}_n^{hf}(\xi_x,y)|\lesssim\int\limits_{\mathbb{R}}|\mathcal{F}_x(\xi_x,y')|dy'\lesssim\left(\int\limits_{\mathbb{R}}(1+y'^2)^s|\mathcal{F}_x(\xi_x,y')|^2dy'\right)^{\frac{1}{2}},$$

where the last bound follows from the Cauchy-Schwarz inequality and $s>\frac{1}{2}$. From the definition of $\hat{r}_n^{hf}(\xi_x,y)$ and $s'>\frac{1}{2}$ it follows that

$$\|\kappa\hat{r}_n^{hf}\|^2_{L^2_{-s,\perp}}\lesssim\int\limits_{|\xi_x|>A}\int\limits_{\mathbb{R}}(1+y'^2)^s|\mathcal{F}_x(\xi_x,y')|^2dy'd\xi_x. \tag{A.10}$$

Step 2.2. Estimating $\hat{r}_n^{lf}(\xi_x,y)$. To estimate $\hat{r}_n^{lf}(\xi_x,y)$, we use the estimate (A.9) for small $|\omega-\omega_n|$ in (A.2) which results in

$$\left|\kappa\hat{r}_n^{lf}(\xi_x,y)\right|\lesssim A|\omega_n-\omega|\int\limits_{\mathbb{R}}(|y|+|y'|)|\mathcal{F}_xf(\xi_x,y')|dy',$$

and using the Cauchy-Schwarz inequality ($s>\frac{3}{2}$) yields

$$\left|\kappa\hat{r}_n^{lf}(\xi_x,y)\right|\lesssim A|\omega_n-\omega|\,(|y|+1)\,\|\mathcal{F}_xf(\xi_x,.)\|_{L^2_s(\mathbb{R})}.$$

Finally, we obtain ($s'>\frac{3}{2}$)

$$\|\kappa\hat{r}_n^{lf}\|^2_{L^2_{-s',\perp}}\lesssim A^2|\omega_n-\omega|^2\|\mathcal{F}_xf\|^2_{L^2_{s,\perp}}. \tag{A.11}$$

Step 2.3. Summary. Combining (A.10), (A.11) yields

$$\|\kappa\hat{r}_n\|^2_{L^2_{-s',\perp}}\lesssim A^2|\omega_n-\omega|^2\|\mathcal{F}_xf\|^2_{L^2_{s,\perp}}+\int\limits_{|\xi_x|>A}\int\limits_{\mathbb{R}}(1+y'^2)^s|\mathcal{F}_x(\xi_x,y')|^2dy'd\xi_x.$$

For any $\varepsilon>0$, we can choose $A:=A_\varepsilon$ so that the last term of the above expression does not exceed $\varepsilon^2/2$; next we choose n so that $A_\varepsilon^2|\omega_n-\omega|^2\|\mathcal{F}_xf\|^2_{L^2_{s,\perp}}<\frac{\varepsilon^2}{2}$, which allows us to conclude that $\|\kappa\hat{r}_n\|_{L^2_{-s',\perp}}\to 0$, as $n\to+\infty$. ∎